# Soft Collinear Effective Theory

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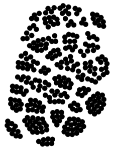
# Effective Theories in general

# Multipole Expansion

Effect of a charge distribution $\rho(\vec{x})$ on test charge q?



*Charge Distribtion*

*Test Charge*

# Multipole Expansion

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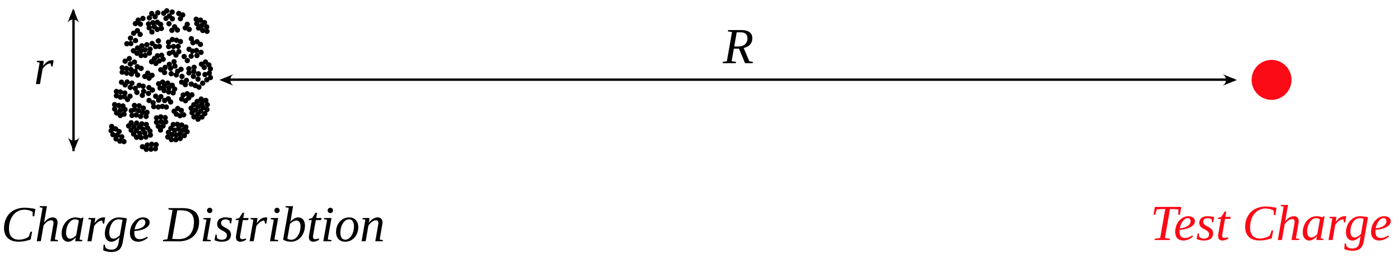



- $r \ll R$: $\rho(\vec{x}) \to$ point charge with $Q = \int d^3x\, \rho(\vec{x})$

# Multipole Expansion

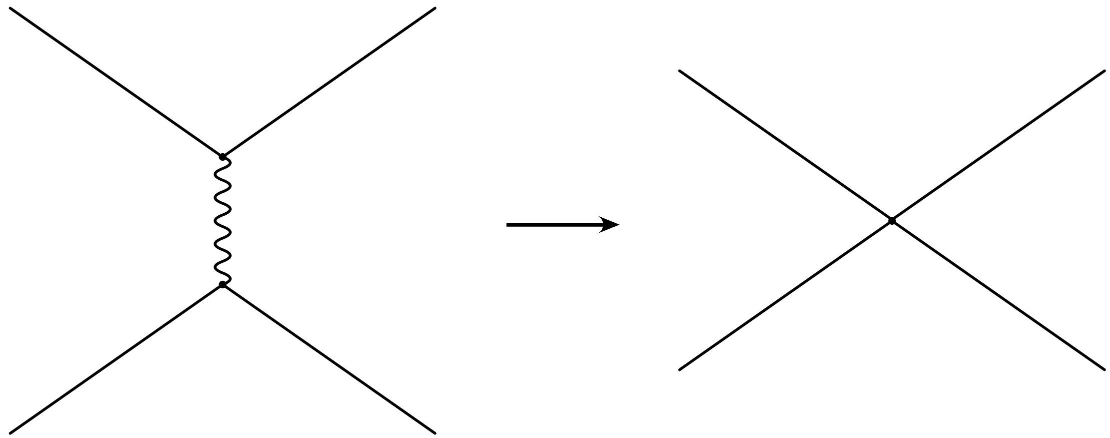
Effect of a charge distribution $\rho(\vec{x})$ on test charge q?


- $r \ll R$: $\rho(\vec{x}) \rightarrow$ point charge with $Q = \int d^3x \, \rho(\vec{x})$
- Corrections: dipol, quadrupole moment of $\rho(\vec{x})$
- Suppressed by powers of $r/R$
- Advantages:
  - Much simpler to calculate
  - Rotational symmetry obvious

# Four-Fermi Interactions

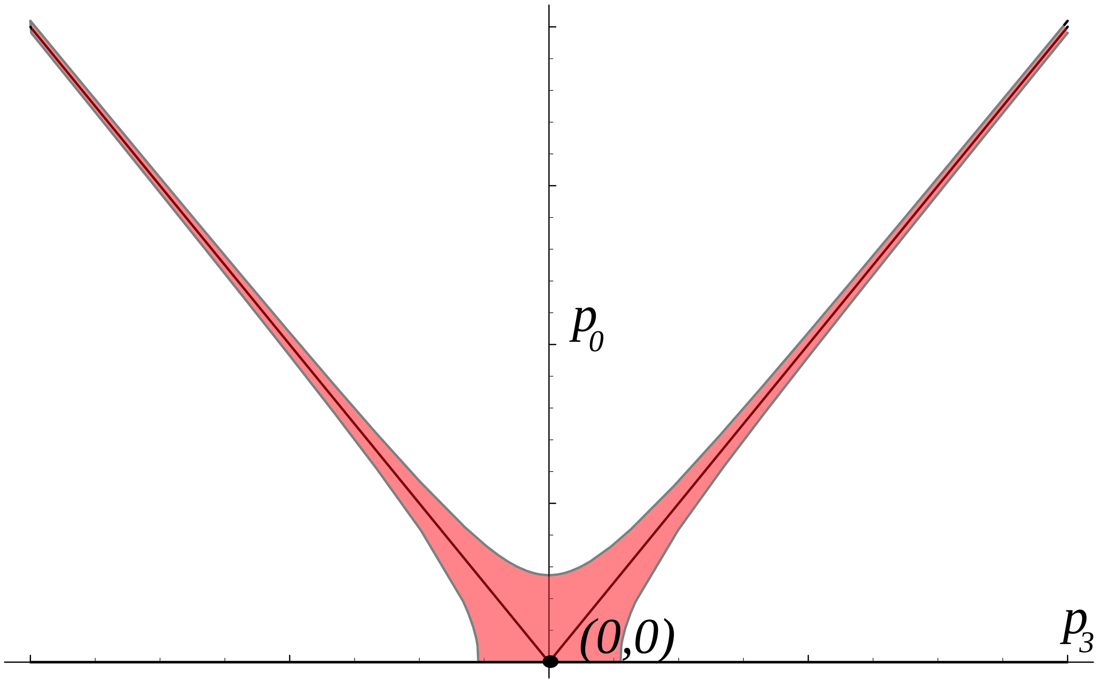
What operators give weak decays?

- Weak interactions mediated by $W$ and $Z$ bosons
- Propagate over distance $r \sim 1/m_W$
- If observed at long distance (low energy $E$), interactions can be treated as local



# Integrating out Fluctuations

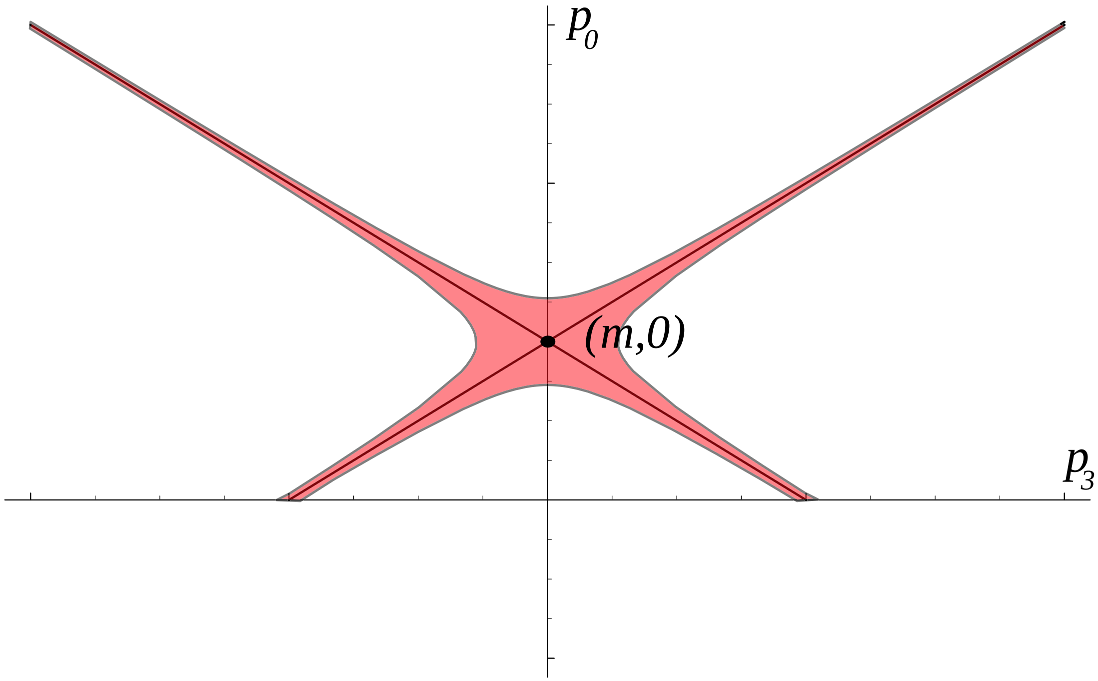
- Observe process at distance scales $R \sim 1/\Lambda$
- Fluctuations at distances $r \ll R$ not relevant
- Integrate out all such fluctuations
- Amounts to integrating out modes with $p^2 > \Lambda^2$



# HQET

- Describe QCD effects in heavy quark physics
- QCD distance scale $R \sim 1/\Lambda_{\rm QCD}$
- Integrate out fluctuations with $p^2 > \Lambda_{\rm QCD}^2$
- Heavy quarks in initial and final states: $p^2 = m_Q^2$

Integrate out fluctuations $(p - m_Q v)^2 > \Lambda_{\rm QCD}^2$



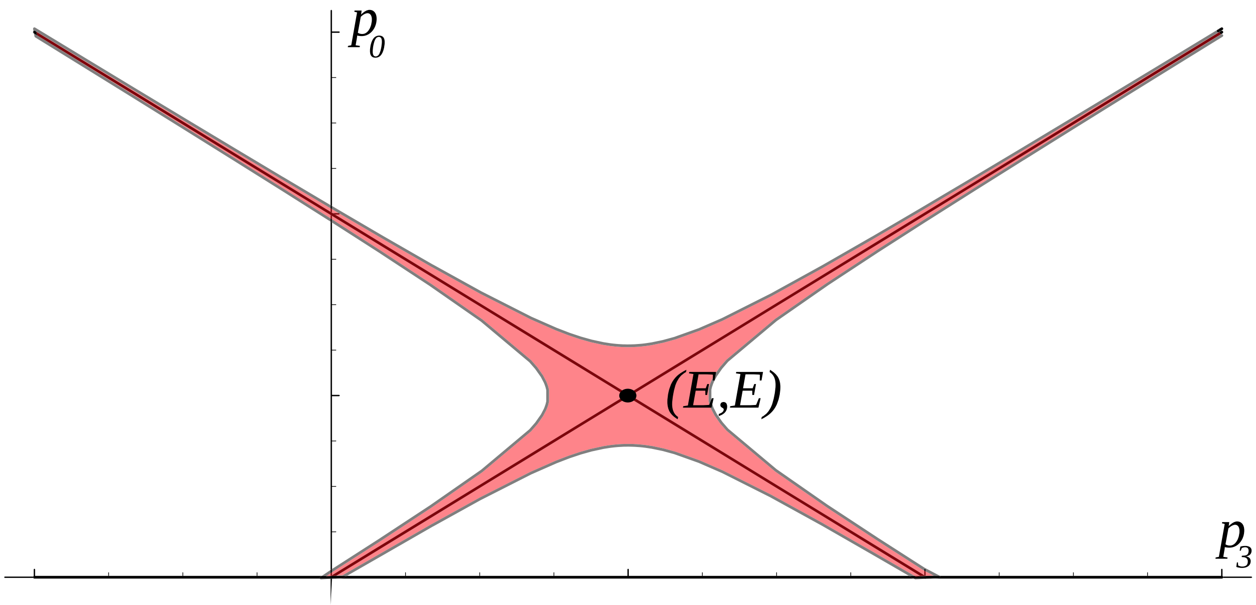
# Ingredients of EFT's

- Identify the degrees of freedom
  - EFT has to reproduce the long distance physics of original theory
- Have a consistent power counting
  - Need a ratio of scales as expansion parameter
- Identify Symmetries
  - Usually Gauge, Lorentz, plus emerging symmetries in low energy limit
- Lagrangian
  - Defines the theory

# SCET

- Identify the degrees of freedom
  - Collinear, soft and usoft fields reproduce infrared of QCD
- Have a consistent power counting
  - Expansion parameter $\lambda^2 = p_\perp^2 / E^2$
- Identify Symmetries
  - Gauge and Lorentz symmetries as well as emerging helicity symmetry
- Lagrangian
  - Given later

# SCET




- The point $(E, E)$ is dynamical and thus not fixed
- Splitting $E \to E_1 + E_2$ has to be described as well
- Very different from Four-Fermi and HQET

# Introduction to SCET

C.B, S. Fleming, M. Luke, Phys. Rev. D63, 014006

C.B, S. Fleming, D. Pirjol, I. Stewart, Phys. Rev. D63, 114020

C.B, I. Stewart, Phys. Lett. B516, 134

C.B, I. Stewart, D. Pirjol, Phys. Rev. D65, 054022

# Scales and DOF

**The relevant scales in the theory**

Two general cases:

a) $E^2 \gg p_c^2 \gg \Lambda_{\rm QCD}^2$ b) $E \gg p_c^2 \sim \Lambda_{\rm QCD}^2$

**The relevant scales in the theory**

LC coordinates:

$$p^\mu = (n\cdot p, \bar{n}\cdot p, p^\perp)$$

$\frac{1}{2}(p^0+p^3)$ $\frac{1}{2}(p^0-p^3)$ $p^i$

| Collinear | Soft | Usoft |
| --- | --- | --- |
| $p \sim Q(\lambda^2, 1, \lambda)$ | $p \sim Q(\lambda, \lambda, \lambda)$ | $p \sim Q(\lambda^2, \lambda^2, \lambda^2)$ |

# Power Counting

## Power counting in the effective theory

- Power counting determined by power of vertices
- Overall power is $\lambda^\delta$ with

BPS('02)

$$\delta = 4u + 4 + \sum_k (k-4)(V_k^c + V_k^s + V_k^{sc}) + (k-8)V_k^{us}$$

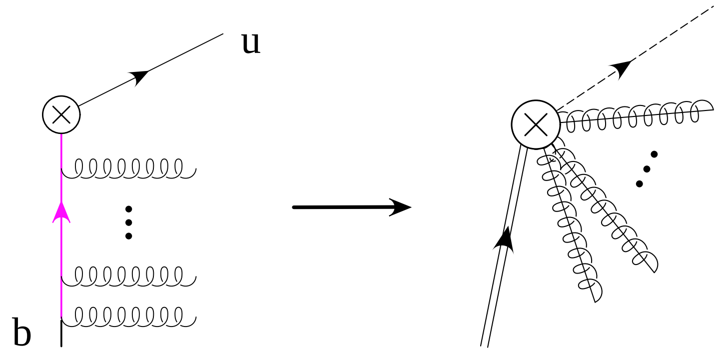
## Scaling of fields in the effective theory

| Type | $(p^+, p^-, p^\perp)$ | Fields | Fermion Scaling |
|---|---|---|---|
| collinear | $(\lambda^2, 1, \lambda)$ | $\xi_{n,p}$, $A_c^\mu$ | $\lambda$ |
| soft | $(\lambda, \lambda, \lambda)$ | $q_s$, $A_s^\mu$ | $\lambda^{3/2}$ |
| usoft | $(\lambda^2, \lambda^2, \lambda^2)$ | $q_{us}$, $A_{us}^\mu$ | $\lambda^3$ |

# Gauge invariance

- Weak currents in QCD are gauge invariant
  $\bar{b}(x)\Gamma u(x) \to \bar{b}(x)U^\dagger(x)\Gamma U(x)u(x) = \bar{b}(x)\Gamma u(x)$
- "Collinear gauge transformation"
  $\partial^\mu U(x) \sim (\lambda^2,\, 1,\, \lambda)$
  $\partial^\mu h_v(x) \to \partial^\mu U(x)h_v(x) \sim (\lambda^2,\, 1,\, \lambda)$

Heavy quarks don't transform under collinear gauge transformation




- Introduce Wilson lines
  $W(x) = \mathsf{P}\exp\left(ig\int_{-\infty}^{x} ds\bar{n}\cdot A_n(s\bar{n})\right) \to U(x)W(x)$
- Gauge invariant combination
  $\bar{\xi}(x)W(x) \to \xi(x)\,U^\dagger(x)U(x)\,W(x) = \bar{\xi}(x)W(x)$

# Collinear Lagrangian

## Leading order [ $\mathcal{O}(\lambda^4)$]

$$\mathcal{L}^{(0)} = \bar{\xi}\left[in\cdot D + i\not{D}_\perp^c \frac{1}{\bar{n}\cdot D_c} i\not{D}_\perp^c\right]\frac{\not{\bar{n}}}{2}\xi \qquad D^\mu = D_c^\mu + igA_{us}^\mu$$

## Subeading order [ $\mathcal{O}(\lambda^5)$]

$$\mathcal{L}^{(1)} = \bar{\xi} i\not{D}_\perp^{us} \frac{1}{\bar{n}\cdot D_c} i\not{D}_\perp^c \frac{\not{\bar{n}}}{2}\xi + \text{h.c.} \qquad D_{us}^\mu = \partial^\mu + igA_{us}^\mu$$

- Subleading Lagrangians known to $\mathcal{O}(\lambda^6)$
  Manohar, Mehen, Pirjol, Stewart, ('02)
  Chay, Kim ('02), Beneke *et al.* ('02)
- Gluon Lagrangian also known
- Usoft and Soft Actions are just like QCD

# Decoupling Usoft Gluons

- Field redefinition: $\xi_{n,p} = Y_n \xi_{n,p}^{(0)}\,, \qquad A_{n,q} = Y_n A_{n,Q}^{(0)} Y_n^\dagger$

$$Y_n(x) = \mathrm{P}\ \exp\left( ig \int_{-\infty}^{x} ds\, n\cdot A_{us}(sn) \right)$$

- Leading Collinear Lagrangian free w/r to usoft interactions

$$\mathcal{L}^{(0)}(\xi_{n,p},\, A_{n,q},\, A_{us}) = \mathcal{L}_c(\xi_{n,p}^{(0)},\, A_{n,q}^{(0)},\, 0)$$

Matrix elements don't involve usoft interactions
All nonperturbative usoft effects are explicit through factors of $Y_n$ in operators

# SCET Summary

- Collinear, soft and usoft degrees of freedom
- Integrating out off-shell fluctuations $\Rightarrow$ Wilson lines $W$ and $S$
- Wilson lines ensure gauge invariance
- Simple power counting from scaling of fields in interaction vertices
- Leading and Subleading Lagrangians known
- Decoupling of usoft gluons by field redefinition with Wilson line $Y$

# Applications

C.B, I. Stewart, D. Pirjol, Phys. Rev. Lett. 87, 201806

C.B, A. Manohar, M. Wise, hep-ph/0212255

# Factorization in $B \to D\pi$

BPS ('01)

Factorization proof in three steps

- Match onto the effective theory

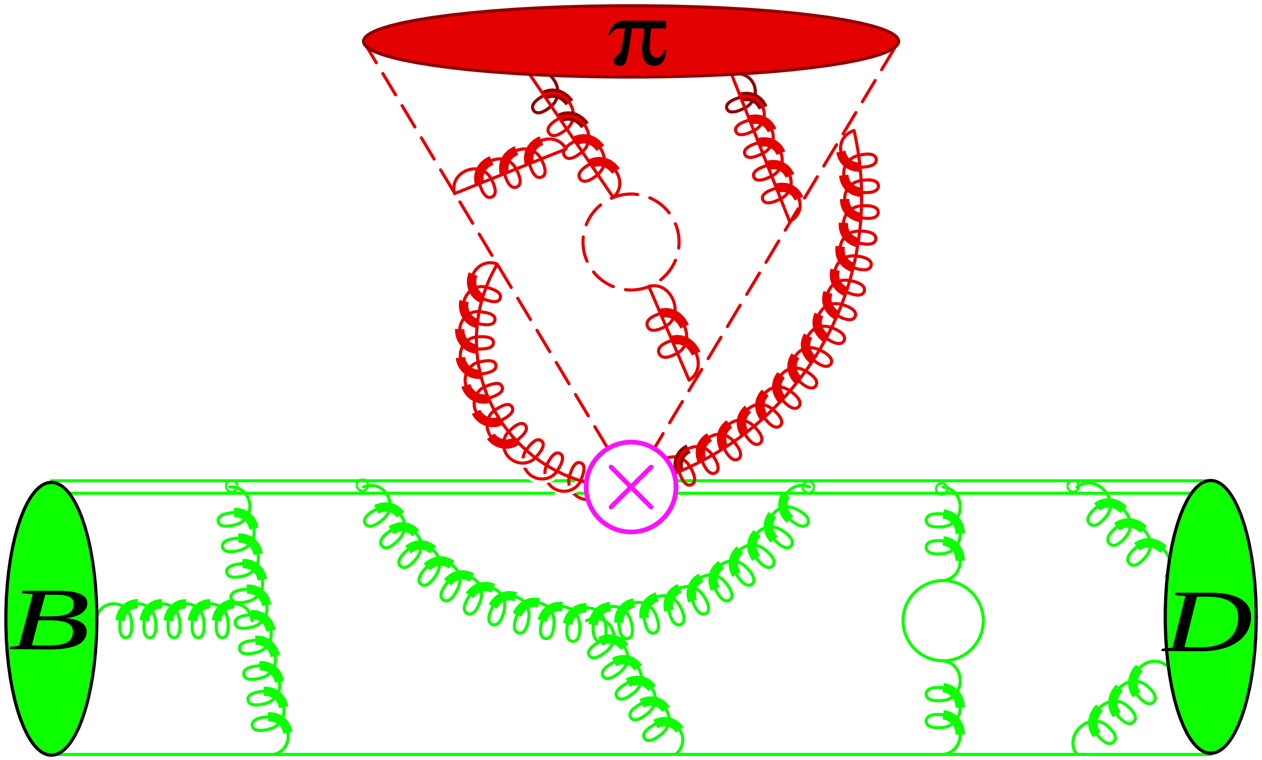
$$\left.\begin{array}{c}[\bar{c}\, b]\,[\bar{u}\, d] \\ [\bar{c}\, T^A b]\,[\bar{u}\, T^A d]\end{array}\right\} \to \left\{\begin{array}{c}[\bar{h}_{v'}^{(c)}\; h_v^{(b)}]\,[\bar{\xi}_{n,p'}^{(d)} W C_{\mathbf{0}}(\bar{\mathcal{P}}_+) W^\dagger \xi_{n,p}^{(u)}] \\ [\bar{h}_{v'}^{(c)} S\, T^A S^\dagger\, h_v^{(b)}]\,[\bar{\xi}_{n,p'}^{(d)} W C_{\mathbf{8}}(\bar{\mathcal{P}}_+) T^A W^\dagger \xi_{n,p}^{(u)}]\end{array}\right.$$

- Decouple usoft gluons by $\xi^{(0)} = Y\xi$
- Take matrix elements

$$\begin{aligned}\langle \pi_n | [\bar{\xi}_{n,\bar{p}}^{(0)} W^{(0)} C_{\mathbf{0}}(\bar{\mathcal{P}}_+) W^{(0)\dagger} \xi_{n,p}^{(0)}] | 0 \rangle &= \frac{i}{2} f_\pi E_\pi \int dx\, C[2E_\pi(2x-1)]\phi_\pi(x) \\ \langle D | \bar{h}_{v'} \Gamma_h h_v | B \rangle &= F^{B\to D}(0)\end{aligned}$$

$$\langle D\pi | \bar{c} b \bar{u} d | B \rangle = N\; F^{B\to D} \int_0^1 dx\, T(x,\mu)\; \phi_\pi(x,\mu)$$

# Factorization in $B \to D\pi$

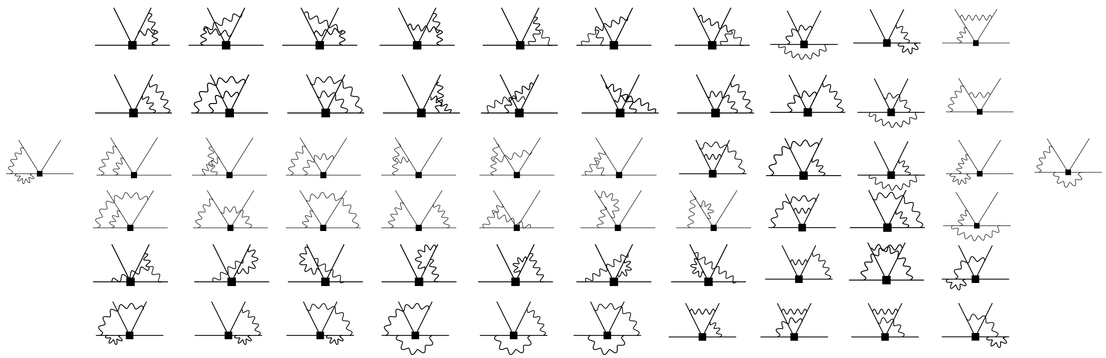


# Factorization in $B \to D\pi$

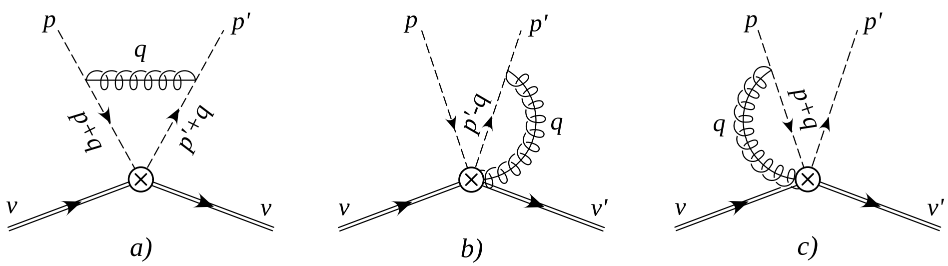
**Factorization at two loops:**

CB, Stewart ('01)

62 2-loop graphs calculated by BBNS (1 hard, 1 collinear)



In the effective theory, all hard in Wilson coefficient
$\Rightarrow$ 3 1-loop graphs in the effective theory




Results agree

# Event shape distributions

CB, Manohar, Wise ('02)
Collins, Korchemsky, Marchesini, Soper, Sterman, . . .

The total decay rate of $Z \to \mathrm{hadrons}$

$$\Gamma = \Gamma^{\mathrm{PT}} \left[ 1 + \frac{\langle \mathrm{Tr}\, G^2 \rangle}{M^4} + \ldots \right]$$

Event shape distributions in endpoint region
$e \sim \Lambda_{\mathrm{QCD}}/M$ (two jet limit)

$$\frac{d\Gamma}{de} = \int de' \, \frac{d\Gamma^{\mathrm{PT}}}{de'} \, S(e - e')$$

For intermediate region $\Lambda_{\mathrm{QCD}}/M \ll e \sim \Delta/M \ll 1$

$$R(\Delta) = \int de \, f(e, \Delta) \frac{d\Gamma}{de} = R^{\mathrm{PT}}(\Delta) \left[ 1 + \frac{\Lambda_{\mathrm{QCD}}}{\Delta} + \frac{\Lambda^2_{\mathrm{QCD}}}{\Delta^2} + \ldots \right]$$

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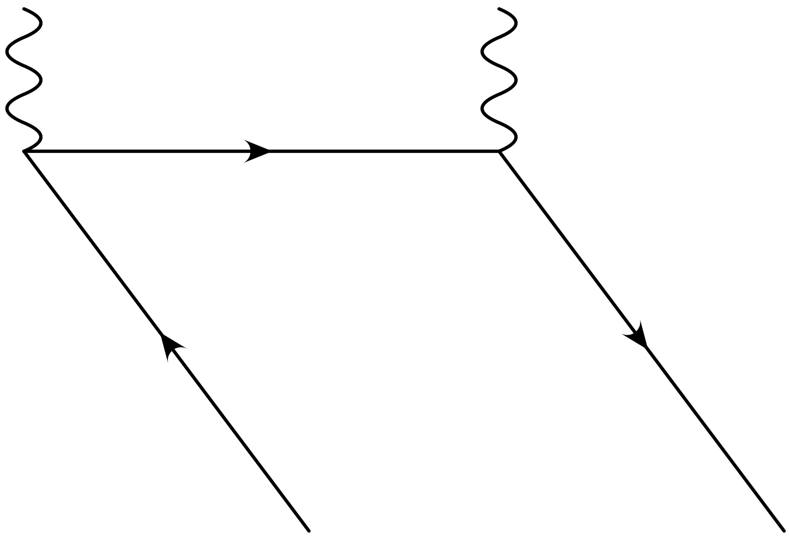
# Strategy

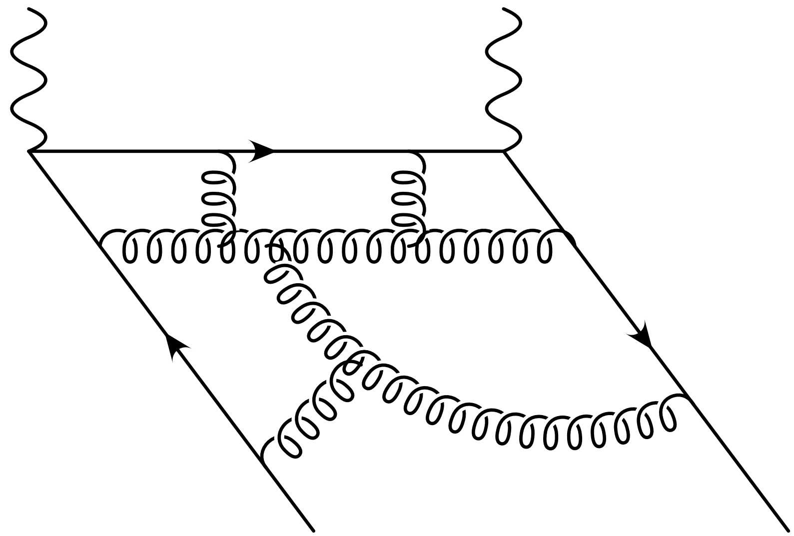
1. Find appropriate differential distributions
2. Derive factorization formula in endpoint region
3. Obtain the enhanced non-perturbative effects from the soft shape function

# Factorization

Parton Model



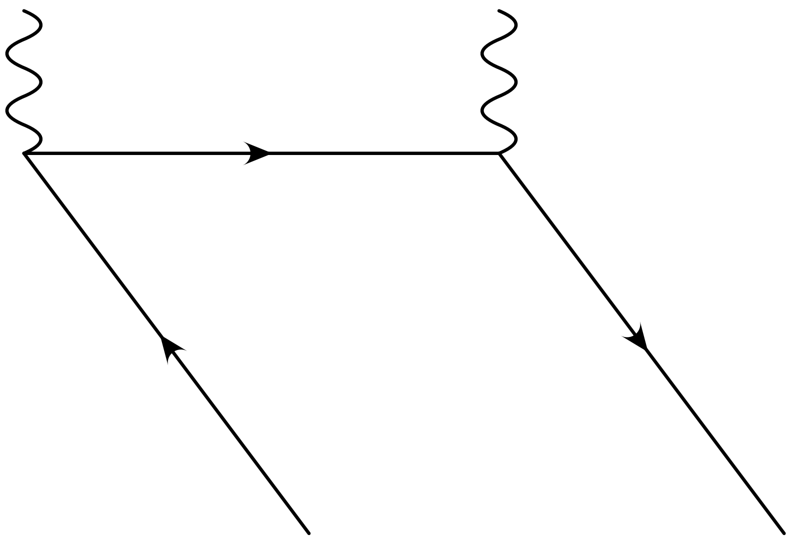
Full QCD

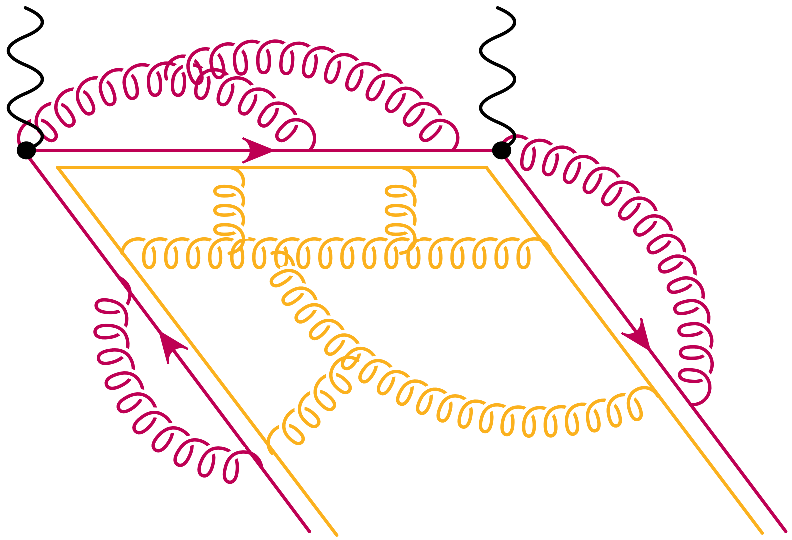


# Factorization

## Parton Model

## SCET



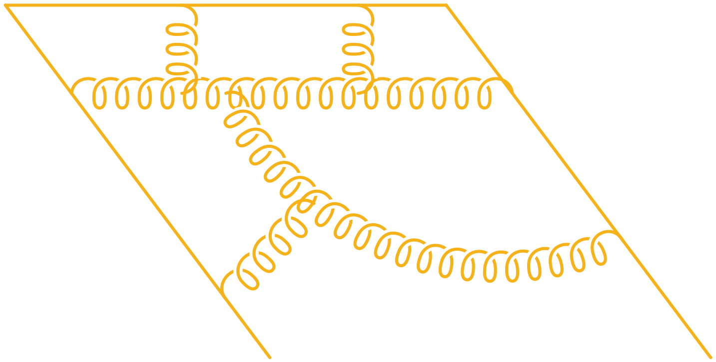


$$\frac{d\Gamma}{dE_J dm_J^2 d\Omega_J} = T(\Omega_J) J(m_J^2/M) \int dk^+ \, J(M - 2E_J - k^+) \, S(k^+)$$

# Smear over region $\Delta$

For $M - 2E_J \sim \Delta \gg \Lambda_{\text{QCD}}$ expand the shape function

$\longleftarrow \quad 1/\Delta \quad \longrightarrow$



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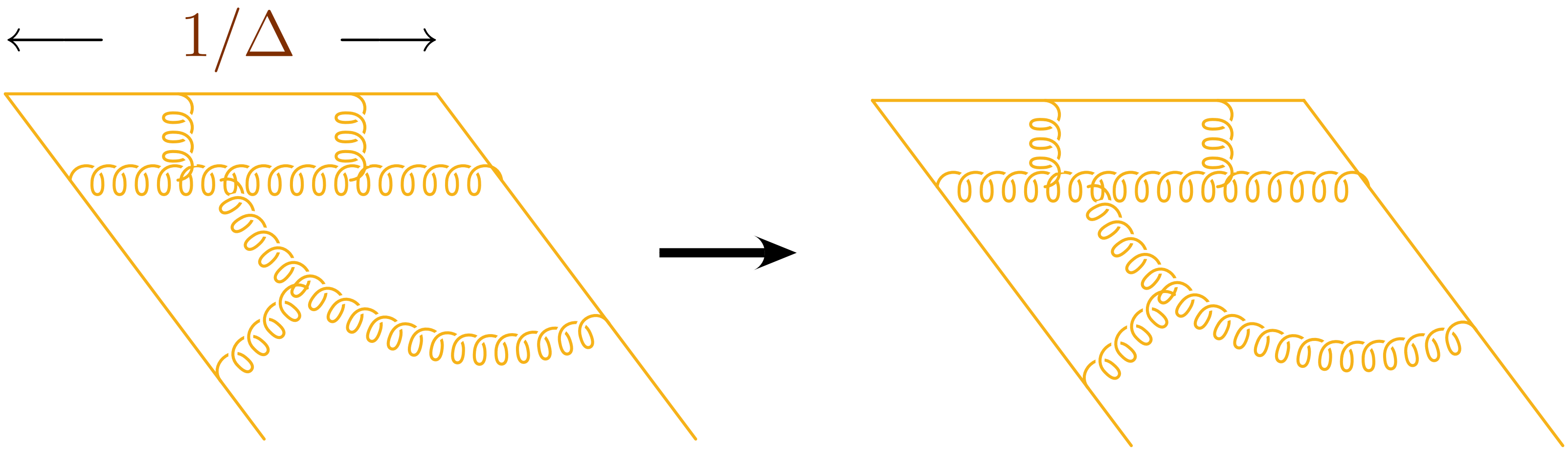

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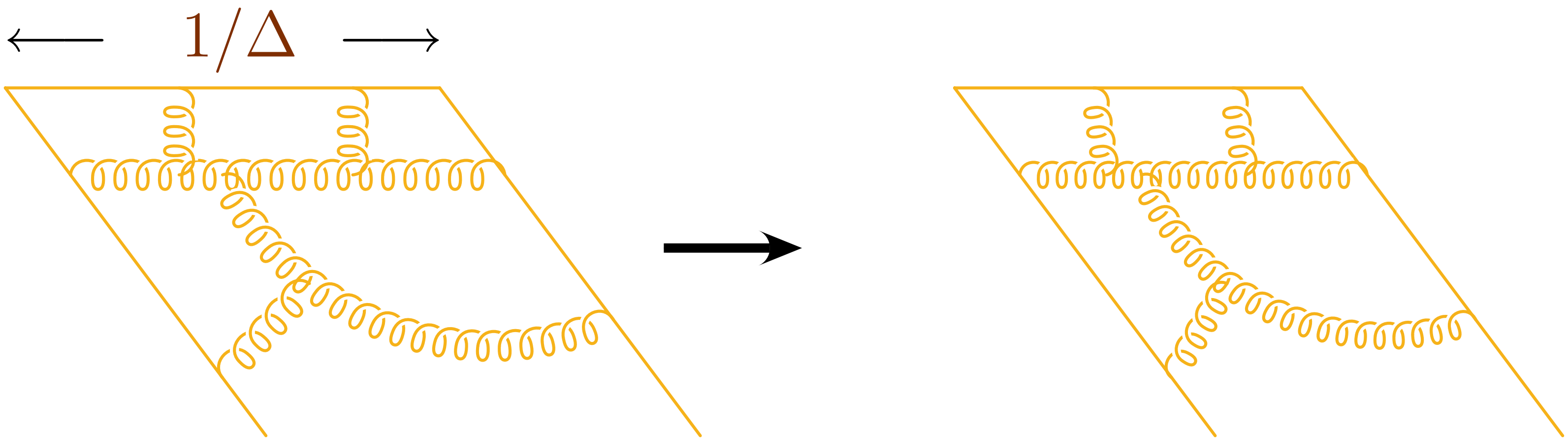

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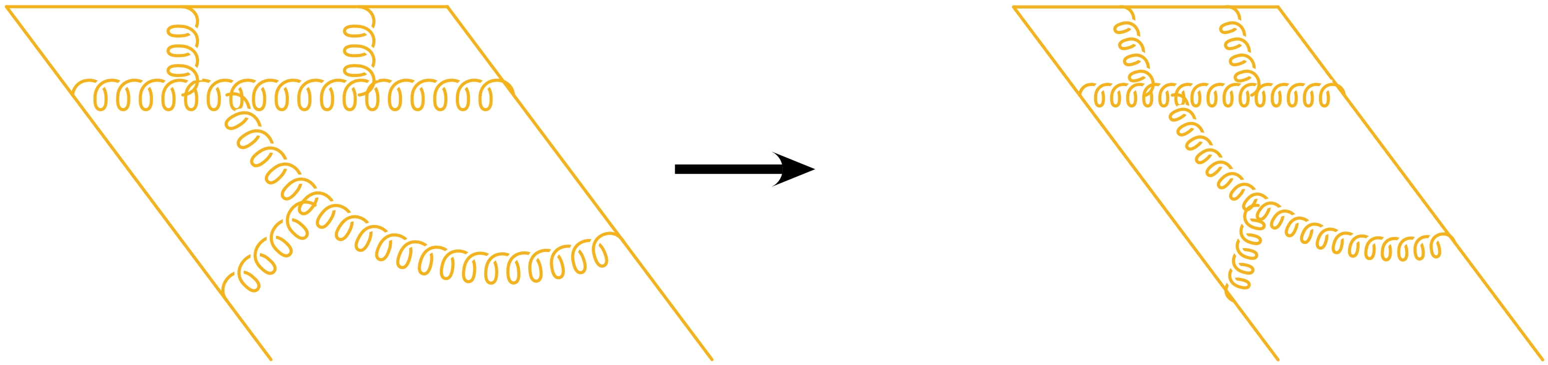

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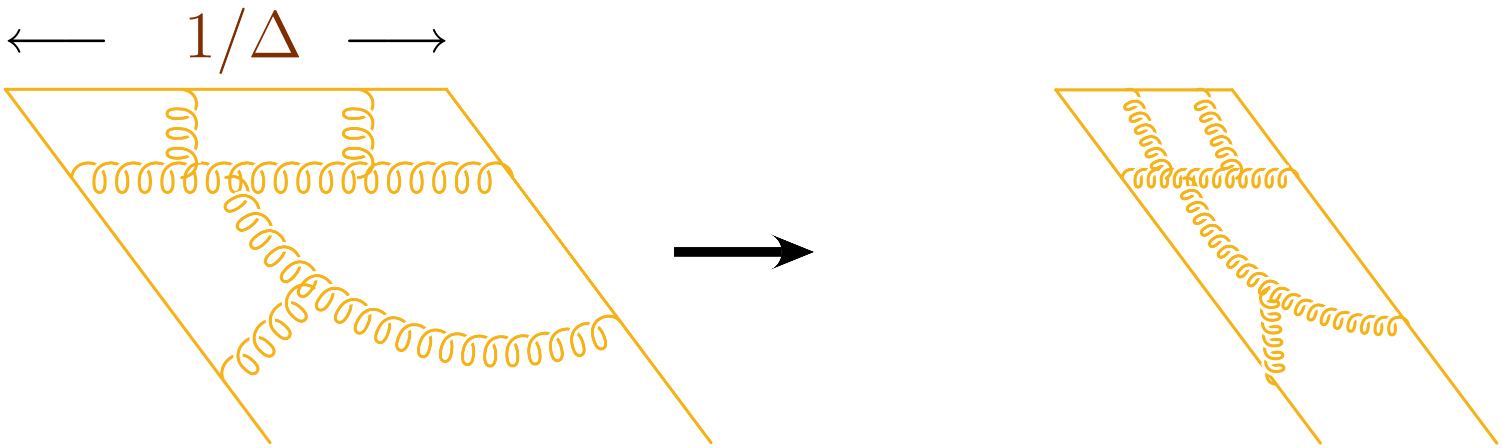

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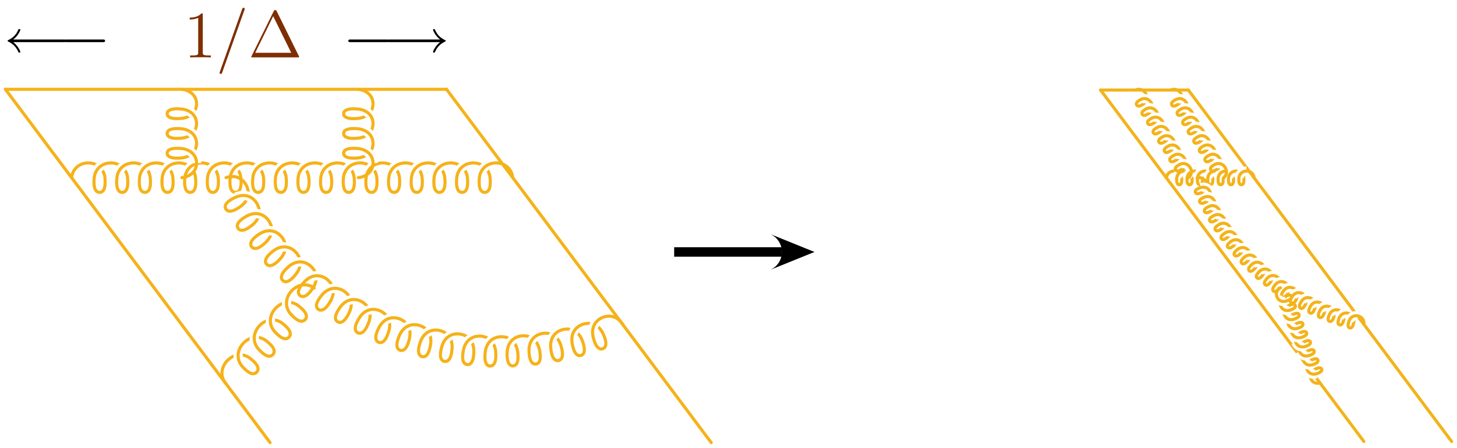

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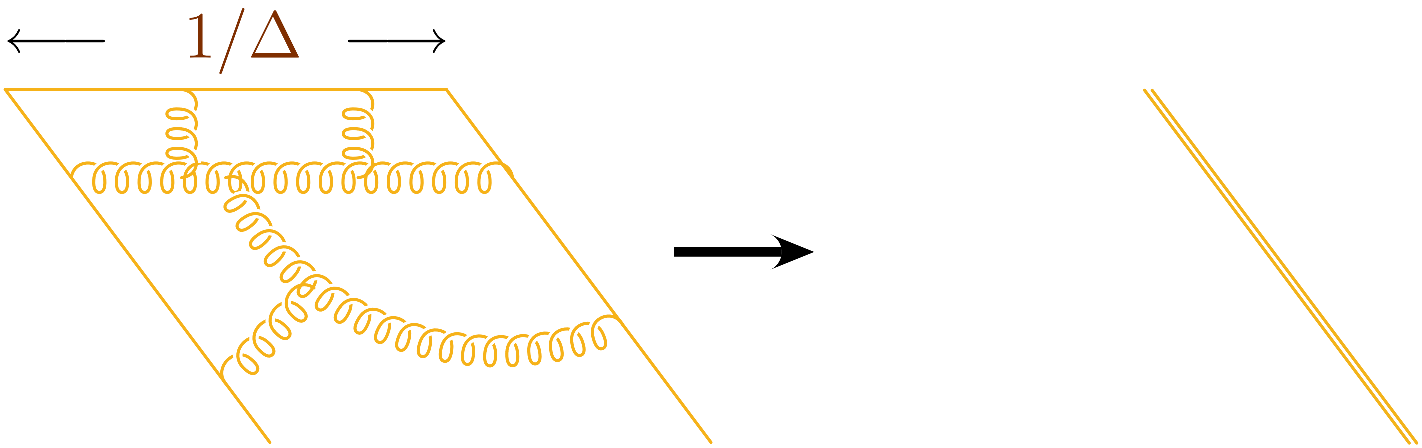

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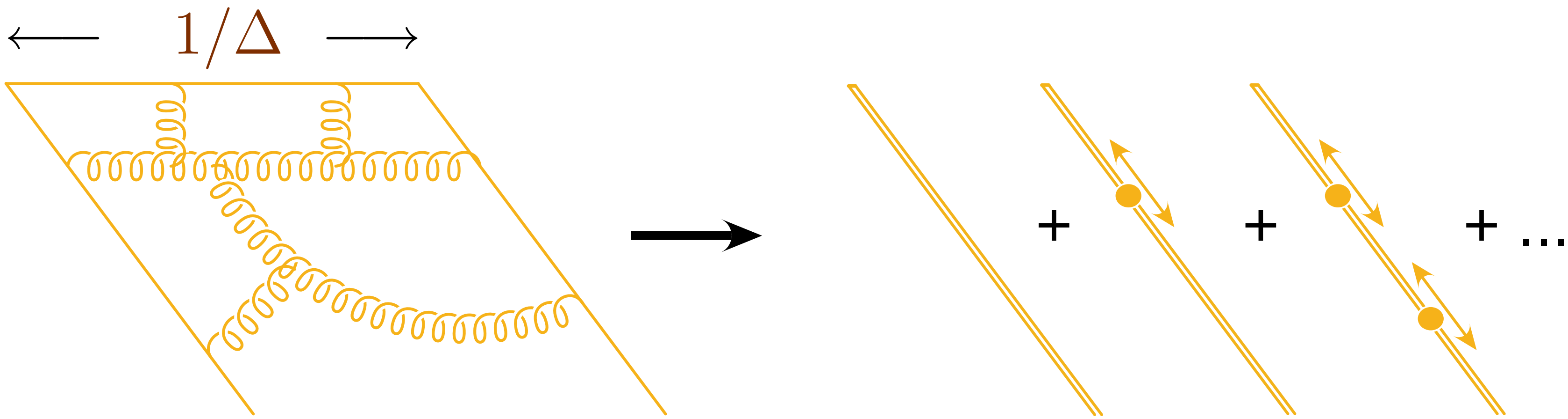

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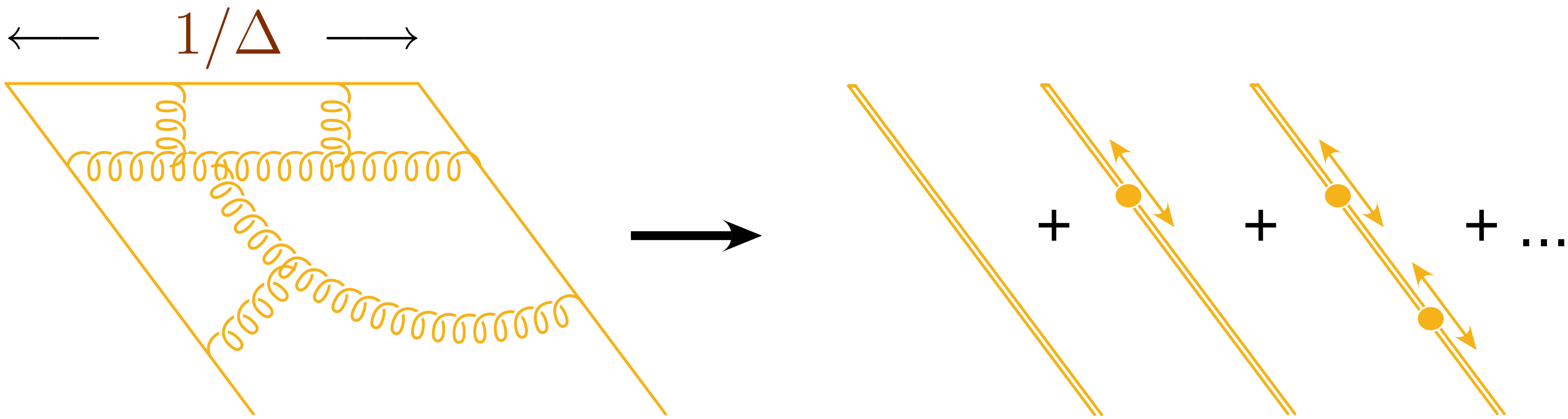



$$S(k) = \frac{1}{3}\left\langle 0 \middle| \mathrm{Tr}\left\{ [Y_{\bar n} Y_n^\dagger]\,\delta(k - in\cdot\partial)[Y_n Y_{\bar n}^\dagger]\right\} \middle| 0 \right\rangle$$

$$= \delta(k) - \underbrace{\delta'(k)\,\langle 0|O^{(1)}|0\rangle}_{\sim \Lambda_{\rm QCD}/\Delta} + \frac{1}{2}\underbrace{\delta''(k)\,\langle 0|O^{(2)}|0\rangle}_{\sim \Lambda^2_{\rm QCD}/\Delta^2} + \ldots$$

$$O^{(m)} = \frac{1}{3}\mathrm{Tr}\left\{[Y_{\bar n} Y_n^\dagger (in\cdot\partial)^m Y_n Y_{\bar n}^\dagger]\right\}, \qquad \langle 0|O^{(m)}|0\rangle = (n\cdot\bar n)^{2m} \mathcal{A}_m^q$$

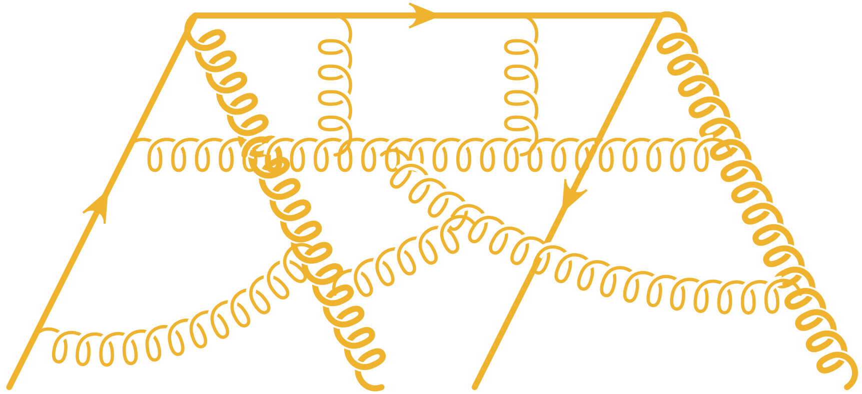
One number ($\mathcal{A}_1^q$): leading non-perturbative

# Three jet events

Similar for three jet events

$$\frac{d\Gamma_{12}}{dE_1 d\Omega_1 dE_2 d\Omega_2}$$

$$= \quad (n_3 \cdot n_2) \frac{d\Gamma_{12}^{(0)}}{dE_1 d\Omega_1 dd\Omega_2} S_{12} \left( M - E_1 n_3 \cdot n_1 - E_2 n_3 \cdot n_2 \right)$$



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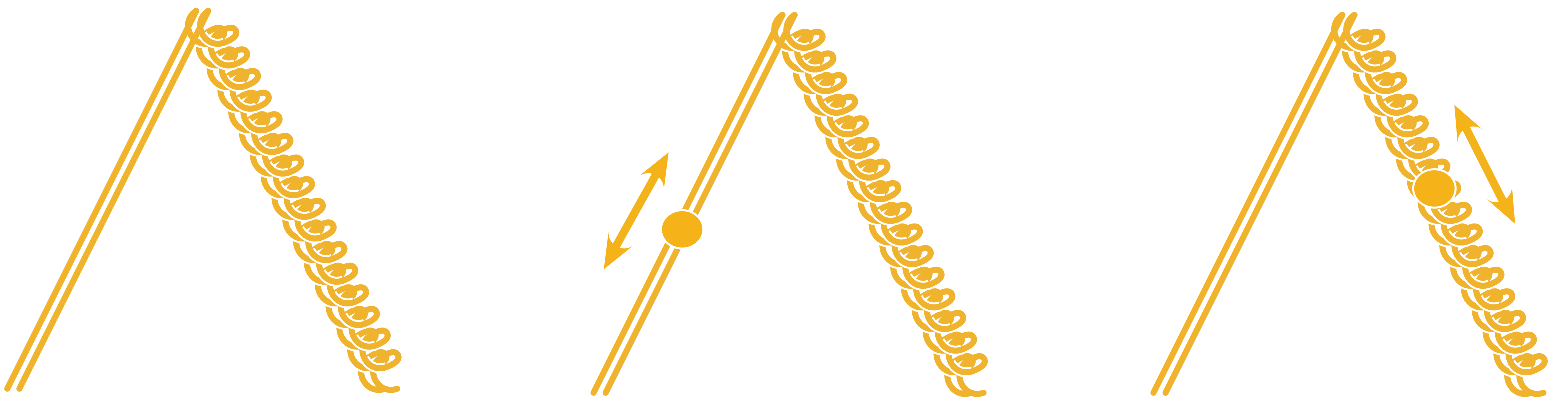
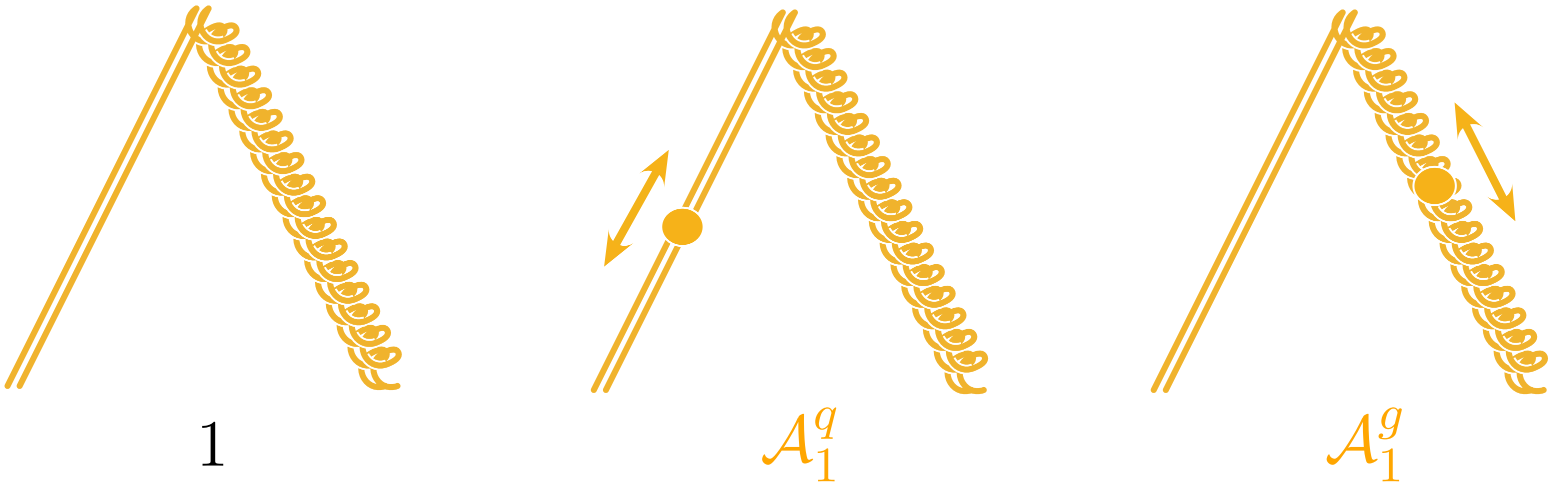


# Three jet events

For $|M - E_1 n_3 \cdot n_1 - E_2 n_3 \cdot n_2| \sim \Delta$

$$\begin{aligned} S_{12}(k) &= \delta(k) - \delta'(k)\, \langle 0|O^{(1)}|0\rangle + \frac{1}{2}\delta''(k)\, \langle 0|O^{(2)}|0\rangle + \ldots \\ &= \delta(k) - \delta'(k)\left[n_3 \cdot n_1 \mathcal{A}_1^q + n_3 \cdot n_2 \mathcal{A}_1^g\right] + \ldots \end{aligned}$$



# Higher number of jets

- A new shape function describes each new distribution in the endpoint
- Enhanced non-perturbative terms only need one insertions
  - Insertion on a quark line $\Rightarrow \mathcal{A}_1^q$
  - Insertion on a gluon line $\Rightarrow \mathcal{A}_1^g$

No new non-perturbative quantities needed for four or more jets

# Summary

- Effective Theories appropriate for processes with widely separated scales
- Reproduces the long distance (low energy) physics
- Expand around different points in Minkowski space
- SCET expands around $(p_0, |\vec{p}|) \sim (E, E)$
- Needs collinear, soft/usoft DOF
- Gauge invariance requires Wilson lines
- Presented factorization proof for $B \to D\pi$
- Presented smeared jet energy distributions
- At leading order two numbers required
- Describe smeared distributions for any number of jets